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The construction of symmetric nose shapes of minimum wave drag☆

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Abstract

The problem of minimizing the wave drag of axisymmetric noses in the supersonic flow of an inviscid, non-heat-conducting perfect gas is considered. A procedure is proposed for constructing the extremal generatrix by step-by-step solution of single parameter problems using assumptions concerning the local relation between the geometric parameters and the gas-dynamic functions. A unique generatrix, from which parts are separated which form noses of arbitrary length, corresponds to the specified free-stream conditions. Comparison of the noses obtained and the noses which are optimal in the exact formulation of the problem shows that the aerodynamic characteristics are close when there is an appreciable difference between the geometric parameters.

One of the ways of improving the aerodynamic characteristics of supersonic aircraft for different purposes involves the construction of axisymmetric noses of low wave drag. The first important results were obtained when solving the problem using a model which employs Newton's formula[1] to relate the geometric and gas-dynamic variables. A number of special features of the optimal shapes was established which, to a considerable degree, determined the direction of subsequent investigations. Experimental confirmation accelerated developments in which the generatrices of bodies which are close to optimal were represented by a power-law dependence of the radius on the longitudinal coordinate.[2] Systematic investigations using the Euler equations showed that the optimal noses have a blunt edge which is a segment of a boundary extremum.[3]

In the Newton model, the generatrices of the optimal noses of all possible aspect ratios are elements of a unique generatrix. It can be constructed using subdivision into segments for which a successive choice of the spatial position is made. First, a vertical segment begins on the axis of symmetry. As the number of segments increases, the broken line adjacent to it converges to a smooth generatrix for which the known necessary condition for an extremum is satisfied (the Euler equation).

This procedure can also be used when the flow is described by a system of Euler equations. The problem of minimizing a function of many variables reduces to a sequence of single parameter problems based on an assumption concerning the local relation between the geometric parameters and the gas-dynamic functions. The position of each segment of the generatrix is chosen without taking account of the segments further downstream. The coordinates of the leading end of a segment are determined by the shape of the initial segment of the generatrix and are therefore fixed. The free parameter is the angle of inclination of a segment with respect to the axis of symmetry. A generatrix

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constructed in this manner, which we shall subsequently call a locally extremal generatrix, specifies the nose shape for any aspect ratios. Unlike the model which uses Newton's formula, in the exact formulation the geometric parameters of the generatrix depend on the free-stream parameters.

1. Construction of the generatrix of a nose of specified length

It is required to find the minimum wave drag of an axisymmetric nose as a function of many variables: $X(r_1, r_2, \ldots, r_n) = \min$. The generatrix of the nose consists of n segments. The radii $r_i (i = 1, \ldots, n)$ of the cross-sections with coordinates x_i serve as the geometric parameters. The values of x_i are not changed during the optimization process. The radius r_1 defines the dimension of the leading edge $x_1 = 0$. The radius of the base r_n is fixed and corresponds to the specified aspect ratio $\lambda = 0.5x_n/r_n$ ($x_n = L$, where L is the nose length).

The problem is solved by the method of direct optimisation, the basis of which is an objective function in quadratic form. The approximating relations are determined by a local analysis of the aerodynamic load distribution with linearization of the relation between the gas-dynamic and the geometric parameters. This method was previously used to construct close to optimal bodies in a flow with an attached shock wave.[4]

Two computational domains were distinguished in order to model the flow around the nose. The overall boundary of the domains lay in a cross-section, remote from the vertex at a distance equal to the diameter of the leading edge. In the neighbourhood of the edge, a computational mesh was constructed in a spherical system of coordinates and the Euler equations were integrated using the Godunov scheme employing establishment in time.[3] The mesh points were placed closer together towards the sharp bend in the contour and 207 mesh points were placed on the surface of the leading part of the nose. In the supersonic flow domain, the calculation was carried out in a cylindrical system of coordinates by the marching method.[5] The equations of motion were integrated using a McCormack scheme. The flow parameters in the initial cross-section were determined from the solution for the first computational domain. In both cases, the discontinuity in the gas-dynamic variables in the leading shock wave was rigorously isolated. In the direction from the nose surface to the shock wave, the number of mesh points was equal to 84.

The optimization investigations were carried out for a nose aspect ratio $\lambda = 1$, 2, 4 and 8 for a free-stream Mach number $M_\infty = 2$, 4. The ratio of the specific heat capacities $\gamma = 1.4$. The number of geometric parameters was varied from $n = 55$ for short noses to $n = 480$ for noses of greater aspect ratio. Close to the leading edge, the nodal cross-sections condensed as given by the relation

$$x_i = 2r_1(s_1^{i-1} - 1)/(s_1^{25} - 1), \quad i = 1, \ldots, 26; \quad s_1 = 1.1$$

For the remaining cross-sections it was assumed that

$$x_i = x_{26} + (x_n - x_{26})(s_2^{i-26} - 1)/(s_2^{n-26} - 1), \quad i = 27, \ldots, n; \quad s_2 = 1.015$$

The drag coefficient is determined as the drag force relative to the velocity head and the nose base area.

The direct optimization method demonstrated a fairly high rate of convergence. The optimization process consisted of four cycles, in each of which the number of direct calculations was varied from 6 to 12. It was therefore required to consider no more than 50 possible configurations in order to determine the optimal nose of a specified aspect ratio. The results obtained agree with the data from systematic investigations[3] both with respect to the minimum drag coefficient and with respect to the parameters determining the nose shape.

With respect to the values of the drag coefficient, the relative difference did not exceed 0.5%. For the leading edge diameter, the divergence was from 2 to 13%. When $M_\infty = 2$, the angle of inclination of the generatrix, φ_1, close to the leading edge increases from 57.4° (when $\lambda = 1$) to 59.2° (when $\lambda = 8$) as the aspect ratio becomes higher. An inverse relationship is observed when $M_\infty = 4$ and, when the elongation is increased, the angle φ_1 decreases from 58.4° to 54.8°. Values of 55° when $M_\infty = 2$ and 56° when $M_\infty = 4$ were obtain earlier for the angle φ_1 in Ref. 3.

2. Construction of a locally extremal generatrix

The assumption that there is a local relation between the geometric parameters and the gas-dynamic functions enables us to simplify the optimization process considerably. In this case, a unique generatrix is constructed for noses of different aspect ratio.

We will now consider a generatrix which begins on the axis of symmetry $x_0 = 0$, $r_0 = 0$ and departs to infinity. The generatrix consists of segments, the first of which defines the leading edge. For convenience, the length of this segment is adopted as the characteristic linear dimension: $r_1 = 1$, $x_1 = 0$. The distribution of the segments in the longitudinal direction is fixed: $x_i = \text{const}$. The ordinates r_i of the ends of the segments served as the independent geometric parameters.

The generatrix determines the shape of axisymmetric nose for any aspect ratio $\lambda_n = 0.5x_n/r_n$ (taking account of a possible change in the longitudinal distribution of the segments). The drag of the nose is represented by the sum:

$$X_n = \pi\sum_{i=1}^{n}(p_i - p_\infty)(r_i^2 - r_{i-1}^2)$$

where p_∞ is the free-stream pressure, p_i is the pressure on a segment of the nose surface corresponding to a segment of the generatrix with number i.

In the exact formulation, the pressure distribution for all segments with numbers greater than $i - 1$ depends on the magnitude of the parameter r_i. The conditions for the nose shape to be optimal are that the derivatives of the drag X_n with respect to the variables r_i ($i = 1, \ldots, n - 1$) are equal to zero. In the case of local flow models, the parameter r_i only determines the pressure for the neighbouring segments p_i and p_{i+1}. The optimality conditions are correspondingly changed. It is necessary to ensure that the derivative of the drag $X_{i+1} - X_{i-1}$ for a pair of neighbouring segments with respect to r_i is equal to zero. Each of the conditions when $i = 1, \ldots, n - 1$ relates a triplet of parameters r_{i-1}, r_i, r_{i+1}. Since r_0 and r_1 are specified, the optimization process reduces to the step-by-step determination of the optimal values of $r_2, \ldots, r_n$ in terms of the solution of one-parameter problems.

The advantages obtained by using local models can also be realized in the exact formulation of the problem. In the case of a local extremal generatrix, the optimality conditions are satisfied assuming the local relation between the gas-dynamic parameters and the nose shape.

If the drag X_2 only depends on the parameter r_1 for the first two segments, then the necessary condition for an extremum is represented by the equality

$$2(p_1 - p_\infty)r_1 - 2(p_2 - p_\infty)r_1 + (\partial p_2/\partial r_1)(r_2^2 - r_1^2) = 0$$

Here, $\partial p_2/\partial r_1$ is the derivative of the pressure for the second segment with respect to the parameter r_1 and p_1 is the mean pressure on the edge, which is independent of r_1. The value which corresponds to the local extremal generatrix is determined as a result of numerical calculations for different r_2.

For pairs of segments with numbers i and $i + 1$ ($i > 1$), we have the condition

$$2(p_i - p_\infty)r_i + (\partial p_i/\partial r_i)(r_i^2 - r_{i-1}^2) - 2(p_{i+1} - p_\infty)r_i + (\partial p_{i+1}/\partial r_i)(r_{i+1}^2 - r_i^2) = 0$$

from which, in the case of known values of r_{i-1} and r_i, the extremal value of r_{i+1} can be found numerically. Hence, a procedure for the step-by-step construction of the segments determining the locally extremal generatrix has been obtained. At each step, the solution of a problem with a single independent variable is found. The assumption of "localness" removed the need to solve the complex problem of searching for a minimum of a function of many variables.

3. Analysis of the results of the investigation

Within the framework of local models of the flow, the locally extremal generatrix is identical to the optimal generatrix. We will now consider the construction of the generatrix for the case when the surface pressure is determined using Newton's formula. The pressure coefficient, corresponding to the segment with the number i, can be represented in the form

$$C_{pi} = 2r_i'^2/(1 + r_i'^2); \quad r_i' = (r_i - r_{i-1})/(x_i - x_{i-1}), \quad i > 0$$

We shall assume a uniform longitudinal distribution of the segments $x_i = (i - 1)\,\Delta x$ ($i > 0$). The optimality conditions are analogous to the Weierstrass-Erdmann condition at a corner point.

When $\Delta x \to 0$, the broken-line generatrix converges to a smooth generatrix adjacent to the leading edge, which corresponds to the exact solution (a Newtonian nose). The number of segments constituting the generatrix varied from

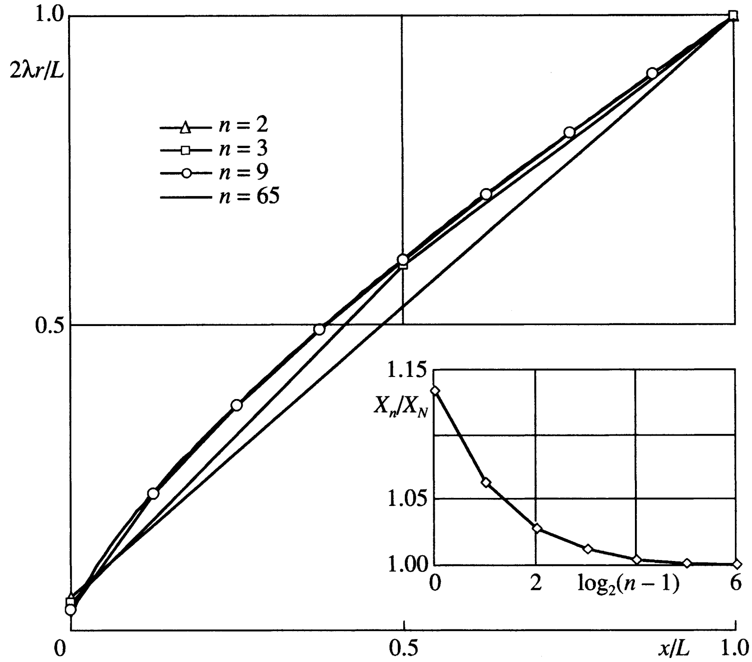

Fig. 1.

2 to 65. In the case when $n = 2$, the optimal nose is a truncated cone. As n increases, the diameter of the leading edge and the drag of the corresponding nose X_n decrease. Four broken-line generatrices, corresponding to different Δx, are shown in Fig. 1 for an elongation $\lambda = 2$. The ratio X_n to the drag X_N of a nose with a smooth generatrix is also shown in this figure as a function of n. Compared with a Newtonian nose, a nose with a broken-line generatrix consisting of five segments has 2.9% higher drag. In the case of a number of segments $n = 65$, the difference is just 0.07%.

In the exact formulation, the flow is described by a system of Euler equations and the dependence of the surface pressure distribution on the geometric parameters is established in a numerical calculation. The locally extremal generatrices are determined for free-stream conditions corresponding to a Mach number $M_\infty = 2, 4$. The construction was continued up to $\lambda = 10$. The results are compared with data for noses which have minimum drag for a given aspect ratio (Section 1). In Figs. 2 and 3, the solid lines represent the dependences for the locally extremal generatrices. The values corresponding to the optimal noses are labelled with markers.

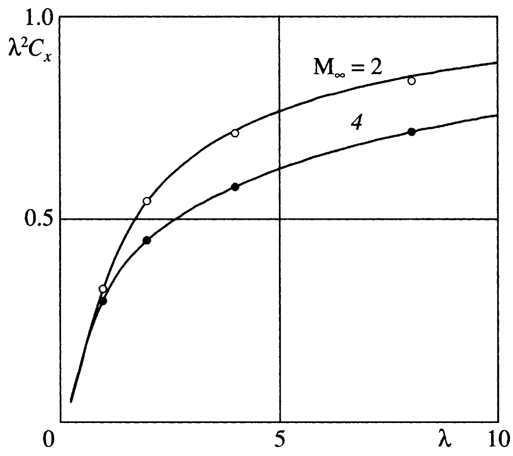

Fig. 2.

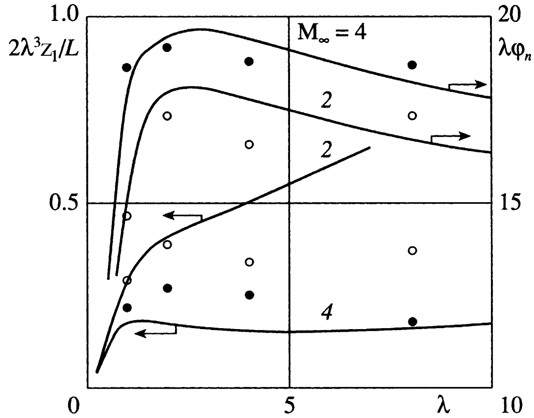

Fig. 3.

The drag coefficient C_x of a nose decreases as the aspect ratio and the Mach number increase, except for the case of low aspect ratios (Fig. 2). The maximum relative excess of the drag of a nose with a locally extremal generatix over the drag of the optimal nose is 1.6% when $M_\infty = 2$ and 1.1% when $M_\infty = 4$. Closeness of the aerodynamic characteristics is attained with a significant difference in the parameters determining the nose shape. The diameter of the leading edge, relative to the diameter of the base, falls rapidly as the elongation increases (Fig. 3). The difference in r_1 for the locally extremal and optimal noses reaches a value of almost 100%. When $M_\infty = 2$, the noses constructed using the local procedure have a leading edge of larger size. On the other hand, when $M_\infty = 4$, larger values of r_1 are obtained for the optimal noses. The angle of inclination, φ_1, of the generatrix near the leading edge depends slightly on the Mach number. Local analysis yields a value $\varphi_1 = 57.6°$. On increasing the aspect ratio, the locally extremal generatrix becomes more mildly sloping. The angle of inclination of the generatrix φ_n at the place where it joins the base decreases as λ increases (Fig. 3). The divergence between the locally extremal and optimal noses does not exceed 10% with respect to this geometric parameter.

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